ON A CLASS OF MINIMUM CONTRAST ESTIMATORS FOR FRACTIONAL STOCHASTIC PROCESSES AND FIELDS

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ABSTRACT. This paper presents the results on consistency and asymptotic normality of a class of minimum contrast estimators for fractional Riesz-Bessel motion based on continuous-time observation. The method does not require discretization, which is necessary in existing approaches. The results are then generalized to random processes and fields with short- or long-range dependence.

1. INTRODUCTION

Many recent studies have found that data in a large number of fields display long-range dependence (LRD) and/or intermittency/volatility clustering (see Beran [15], Frisch [21], Anh and Heyde [3], Leonenko [31], Shiryaev [37] among others). A fundamental LRD process is fractional Brownian motion (FBM), originally introduced by Kolmogorov [30] within a Hilbert space framework and popularized by Mandelbrot and van Ness [35]. A modern theory of FBM can be found in Samorodnitsky and Taqqu [36]. This is a Gaussian process which has stationary increments and spectral density of the form

$$g(\lambda) = \frac{c}{|\lambda|^{2H+1}}, \quad c > 0, 0 < H < 1, \lambda \in \mathbb{R}^1, \tag{1.1}$$

where (1.1) should be understood in the sense of time-scale analysis (Flandrin [19]) or in a limiting sense (Solo [38]) since FBM is a nonstationary process.

A stationary process which displays LRD is the fractional Riesz-Bessel motion (FRBM) introduced in Anh *et al.* [2] (see also Anh and Nguyen [9], Anh *et al.* [7]). This is a stationary Gaussian process $Y(t)$ with spectral density of the form

$$f(\lambda) = f(\lambda, \theta) = \frac{\eta}{|\lambda|^{2\beta}\left(1+\lambda^2\right)^{\alpha}}, \quad \lambda \in \mathbb{R}^1, \tag{1.2}$$

where θ is the unknown parameter vector $\theta = (\alpha, \beta, \eta)' \in \Theta$, Θ being a compact subset of $\left[\frac{1}{2}, \infty\right) \times \left(0, \frac{1}{2}\right) \times (0, \infty)$. The spectral density (1.2) behaves as $O\left(|\lambda|^{-2\beta}\right)$ as $|\lambda| \to 0$ and as $O\left(|\lambda|^{-2(\beta+\alpha)}\right)$ as $|\lambda| \to \infty$. The exponent β determines the LRD of FRBM, while the exponent $\alpha + \beta$ is a fractal index, which indicates the degree of fractality of a path. In fact, the order $O\left(|\lambda|^{-2(\beta+\alpha)}\right)$ as $|\lambda| \to \infty$ will specify the Hausdorff dimension of the path via an Abelian-Tauberian-type theorem (Bingham [16], Adler [1], p. 204). Part of this paper considers the parameter estimation of FRBM with spectral

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density (1.2) based on a continuous-time observation $\{Y(t), 0 \leq t \leq T\}$. Different types of discretization such as instantaneous sampling, increasing domain asymptotics or fixed-domain asymptotics (Leonenko and Woyczynski [34]) are known to lead to possible loss of information on certain fractional parameters of the corresponding spectral density (Stein [39]).We consider a continuous-time approach in this paper. We use some ideas on minimum contrast estimators due to Ibragimov [29] and Leonenko and Moldavs'ka [32]. Nevertheless these two papers dealt with short-range dependent processes, while we present a method for stochastic processes with LRD and fractality. Note that the parameter estimation problem for FRBM has been studied by Anh *et al.* [4] using a wavelet method, and Gao *et al.* [23], [22] using a Whittle-type method with discretization.

The method of this paper provides another type of minimum contrast estimators. Its distinct advantage is that our minimum contrast functional is a linear function of the periodogram. This is significant because a non-linear function of the periodogram leads to the necessity to weight the periodogram (that is, to consider a kernel estimate of the spectral density; see, for example, Dahlhaus and Wefelmeyer [18]). Thus, our functional is another example of an information distance function (the first being the Whittle distance function) which is linear with respect to the periodogram.

Section 2 contains the results on consistency and asymptotic normality of our minimum contrast estimator for fractional Riesz-Bessel motion. In fact, our approach is general and can be applied to stochastic processes and fields with short- or long-range dependence. Section 3 will present the results on consistency and asymptotic normality in a general context for stochastic processes and fields based on continuous-time observations. We also note an application of these results to a class of random processes arising as limits of the rescaled solutions of the heat equation with random long-range dependent data (Anh and Leonenko [5, 6] or Anh *et al.* [8]). Section 4 contains the proofs of these results which are based on some previous results of Bentkus [11], [12], Bentkus and Rutkauskas [14] regarding Fejér-type multidimensional kernels. Our results should be compared with previous results on estimation of unknown parameters of spectral densities (see Whittle [41], Fox and Taqqu [20], Heyde and Gay [27, 28], Giraitis and Taqqu [24], Leonenko and Woyczynski [33, 34], and their references). In fact, all these papers deal with discrete-time processes or discretized data. They are concerned mainly with Whittle-type minimum contrast estimators for short-range or long-range dependent models under Gaussian subordination or linear-type processes. This present paper takes the first step towards a general method for parameter estimation of continuous-time stochastic processes and fields, which possibly possess long-range dependence, based on continuous-time observations . The method, in principle, is applicable to more general non-Gaussian processes and fields and for higher-order spectral densities. We address this approach for higher-order spectral densities in a subsequent paper.

2. Estimation of fractional Riesz-Bessel motion

We first introduce the definition of minimum contrast estimators following Guyon [25], pp. 119-127, where these estimators have been studied for some classes of discrete-time random fields.

Let a random field $Y(t)$, $t \in \mathbb{R}^n$, be observed on a sequence D_T of increasing finite domains, for example, over a parallelepiped $\Pi(T) = \{t \in \mathbb{R}^n : 0 \leq t_i \leq T_i,\ i = 1, ..., n\}$, $T = (T_1, ..., T_n) \in \mathbb{R}^n$. Denote $\tilde{T} = \min\{T_i,\ i = 1, ..., n\}$. Consider a parametric statistical model with a family of distributions $\{P_\theta, \theta \in \Theta\}$, where Θ is a compact subset of $\mathbb{R}^p$ and the true value of the parameter vector $\theta_0 \in int\ \Theta$, the interior of Θ. Denote $P_0 = P_{\theta_0}$.

A nonrandom real-valued function $K(\theta_0;\theta) \geq 0$ is called a *contrast function* if it has a unique minimum at $\theta = \theta_0$. A random field $U_T(\theta)$, $T \in \mathbb{R}^n$, $\theta \in \Theta$, related to the observation $\{Y(t), t \in \Pi(T)\}$ is called the *contrast field* for a contrast function $K(\theta_0;\theta)$ if it satisfies the following inequality:

$$\liminf_{\tilde{T}\to\infty} [U_T(\theta) - U_T(\theta_0)] \geq K(\theta_0;\theta) \qquad \forall\theta \in \Theta \tag{2.1}$$

in P_0-probability. The *mimimum contrast estimator* $\hat{\theta}_T$ is defined as a minimum point of the functional $U_T(\theta)$, that is,

$$\hat{\theta}_T = \arg\min_{\theta\in\Theta} U_T(\theta). \tag{2.2}$$

In the following we denote the conditions by the letters **A**,**B**,**C**, and so on.

A. Let $Y(t)$ be a real-valued measurable stationary Gaussian process with zero mean and spectral density (1.2), observed on an interval $0 \leq t \leq T$. We assume that the true value θ_0 of the parameter vector θ is such that $\theta_0 \in int\ \Theta$.

Note that $f(\lambda,\theta) \not\equiv f(\lambda,\theta')$ for $\theta \neq \theta'$, almost everywhere in $\mathbb{R}^1$, that is, our model is identifiable. Let us consider the following factorization of the spectral density (1.2):

$$f(\lambda,\theta) = \sigma^2(\theta)\,\psi(\lambda,\theta), \quad \lambda \in \mathbb{R}^1, \theta \in \Theta, \tag{2.3}$$

where

$$\sigma^2(\theta) = \int_{\mathbb{R}^1} f(\lambda,\theta)\, w_{a,b}(\lambda)\, d\lambda, \tag{2.4}$$

$$\int_{\mathbb{R}^1} \psi(\lambda,\theta)\, w_{a,b}(\lambda)\, d\lambda = 1, \tag{2.5}$$

and the weight function is chosen to be

$$w_{a,b}(\lambda) = \frac{|\lambda|^{2b}}{\left(1+\lambda^2\right)^a}, \quad \lambda \in \mathbb{R}^1, a > b \geq 0. \tag{2.6}$$

In the case under consideration, we have in fact that the multiplicative parameter η of the spectral density (1.2) is included only in $\sigma^2(\theta) = \sigma^2(\alpha,\beta,\eta)$ and the function $\psi(\lambda;\theta)$ does not depend on η, that is, $\psi(\lambda;\theta) = \psi(\lambda;\alpha,\beta)$. Note that the case $b = 0$, $a = 1$ was considered in Leonenko and Moldavs'ka [32] for random fields with spectral density $f(\lambda,\theta) \in L_2(\mathbb{R}^n)$.

Direct calculations (see Anh *et al.* [7] for details) give the exact form of the function $\sigma^2(\theta)$ as

$$\sigma^2(\theta) = \eta B\left(\frac{1}{2} - \beta + b, \alpha + a - \frac{1}{2} + \beta - b\right), \tag{2.7}$$

where $a > b > 0$.

We will use the following condition

$$\nabla_\theta \int_{\mathbb{R}^1} \psi(\lambda,\theta)\, w_{a,b}(\lambda)\, d\lambda = \int_{\mathbb{R}^1} w_{a,b}(\lambda)\, \nabla_\theta \psi(\lambda,\theta)\, d\lambda = 0, \tag{2.8}$$

where $\nabla_\theta = \left(\frac{\partial}{\partial\alpha}, \frac{\partial}{\partial\beta}\right)$ (the function $\psi(\lambda,\theta)$ does not depend on η). We consider the following constrast process:

$$U_T(\theta) = -\int_{\mathbb{R}^1} I_T(\lambda)\, w_{a,b}(\lambda) \log\psi(\lambda,\theta)\, d\lambda, \tag{2.9}$$

where

$$I_T(\lambda) = \frac{1}{2\pi T}\left|\int_0^T Y(t)\, e^{-it\lambda} dt\right|^2, \quad \lambda\in\mathbb{R}^1$$

is the periodogram of $Y(t)$. We introduce also the function

$$U(\theta) = -\int_{\mathbb{R}^1} f(\lambda,\theta_0)\, w_{a,b}(\lambda)\log\psi(\lambda,\theta)\, d\lambda, \tag{2.10}$$

and the function

$$K(\theta_0;\theta) = \int_{\mathbb{R}^1} f(\lambda,\theta_0)\, w_{a,b}(\lambda)\log\frac{\psi(\lambda,\theta_0)}{\psi(\lambda,\theta)} d\lambda. \tag{2.11}$$

Our first statement is the following.

Theorem 1. *Let the condition A hold and the parameters of the weight function (2.6) satisfy $b > 1/2, a > b + 1/2$. Then the function $K(\theta_0;\theta)$ defined by (2.11) is the contrast function for the contrast process $U_T(\theta)$ defined by (2.9). Moreover the minimum contrast estimator*

$$\widehat{\theta}_T = \left(\widehat{\alpha}_T, \widehat{\beta}_T\right) = \arg\min_{\theta\in\Theta} U_T(\theta) \tag{2.12}$$

is a consistent estimator of the parameters (α,β) as $T \longrightarrow \infty$ and the estimator

$$\widehat{\sigma}_T^2 = \int_{\mathbb{R}^1} I_T(\lambda)\, w_{a,b}(\lambda)\, d\lambda \tag{2.13}$$

is a consistent estimator of the parameter $\sigma^2(\theta)$ as $T \longrightarrow \infty$.

Remark 1. *The estimators (2.12) and (2.13) give an estimator of the parameter η in the following form:*

$$\widehat{\eta}_T = \widehat{\sigma}_T^2\left\{B\left(\frac{1}{2} - \widehat{\beta}_T + b, \widehat{\alpha}_T + a - \frac{1}{2} + \widehat{\beta}_T - b\right)\right\}^{-1}. \tag{2.14}$$

The second result presents the asymptotic normality of the minimum contrast estimators described in Theorem 1. We first introduce the following matrices:

$$S(\theta) = \{s_{ij}(\theta)\}_{i,j=1,2} \tag{2.15}$$

and

$$A(\theta) = \{a_{ij}(\theta)\}_{i,j=1,2} \tag{2.16}$$

with the elements

$$\begin{aligned} s_{ij}(\theta) &= \int_{\mathbb{R}^1} f(\lambda,\theta)\frac{\partial^2}{\partial\theta_i\partial\theta_j}\log\psi(\lambda,\theta)\, w_{a,b}(\lambda)\, d\lambda \\ &= \sigma^2(\theta)\int_{\mathbb{R}^1} w_{a,b}(\lambda)\left[\frac{\partial^2}{\partial\theta_i\partial\theta_j}\psi(\lambda,\theta) - \frac{1}{\psi(\lambda,\theta)}\frac{\partial}{\partial\theta_i}\psi(\lambda,\theta)\frac{\partial}{\partial\theta_j}\psi(\lambda,\theta)\right] d\lambda, \end{aligned} \tag{2.17}$$

$$a_{ij}(\theta) = 4\pi \int_{\mathbb{R}^1} f^2(\lambda,\theta)\, w_{a,b}^2(\lambda) \frac{\partial}{\partial\theta_i} \log\psi(\lambda,\theta) \frac{\partial}{\partial\theta_j} \log\psi(\lambda,\theta)\, d\lambda \tag{2.18}$$

$$= 4\pi \left(\sigma^2(\theta)\right)^2 \int_{\mathbb{R}^1} w_{a,b}^2(\lambda) \frac{\partial}{\partial\theta_i}\psi(\lambda,\theta) \frac{\partial}{\partial\theta_j}\psi(\lambda,\theta)\, d\lambda,$$

$$i,j = 1,2, \theta_1 = \alpha, \theta_2 = \beta.$$

Theorem 2. *Let the condition A hold, and the parameters a, b of the weight function (2.6) satisfy $b > 1$, $a > b + 5/4$, and the matrices $S(\theta)$ and $A(\theta)$ be positive definite. Then, as $T \longrightarrow \infty$, the vector of minimum contrast estimators $\widehat{\theta}_T = \left(\widehat{\alpha}_T, \widehat{\beta}_T\right)$ defined by (2.12) is asymptotically normal with*

$$\sqrt{T}\left(\widehat{\theta}_T - \theta_0\right) \xrightarrow{D} \mathcal{N}_2\left(0, S^{-1}(\theta_0)\, A(\theta_0)\, S^{-1}(\theta_0)\right), \tag{2.19}$$

where the matrices $S(\theta_0)$ and $A(\theta_0)$ are defined by the formulae (2.17) and (2.18) respectively and $\theta_0 = (\alpha_0, \beta_0)$.

Remark 2. *The parameter η in (1.2) can be estimated, in principle, by using the correlogram of the continuous-time observation (see Leonenko* [31]*, Chapter 5). This parameter can also be estimated by using the estimator $\widehat{\sigma}_T^2$ of $\sigma^2(\theta)$ (see (2.12) - (2.14)); the asymptotic normality of the resulting estimator can then be deduced by applying the delta method (see Barndorff-Nielsen and Cox* [10]*, Sections 2.4, 2.7 and 2.9).*

Remark 3. *The elements of the matrix $S(\theta_0)$ can be obtained in the following form:*

$$\begin{aligned}
s_{11}(\theta_0) &= \sigma^2(\theta_0)\,[\left\{\int_{\mathbb{R}^1} \psi(\lambda,\theta_0) \ln\left(1+\lambda^2\right) w_{a,b}(\lambda)\, d\lambda\right\}^2 \\
&\quad - \int_{\mathbb{R}^1} \psi(\lambda,\theta_0) \left\{\ln\left(1+\lambda^2\right)\right\}^2 w_{a,b}(\lambda)\, d\lambda], \\
s_{2,2}(\theta_0) &= \sigma^2(\theta_0)\,[\left\{\int_{\mathbb{R}^1} \psi(\lambda,\theta_0) \ln\left(\lambda^2\right) w_{a,b}(\lambda)\, d\lambda\right\}^2 \\
&\quad - \int_{\mathbb{R}^1} \psi(\lambda,\theta_0) \left\{\ln\left(\lambda^2\right)\right\}^2 w_{a,b}(\lambda)\, d\lambda], \\
s_{1,2}(\theta_0) &= s_{2,1}(\theta_0) = \sigma^2(\theta_0)\,[\int_{\mathbb{R}^1} \psi(\lambda,\theta_0) \ln\left(1+\lambda^2\right) w_{a,b}(\lambda)\, d\lambda
\end{aligned}$$

$$\times \int_{\mathbb{R}^1} \psi(\lambda,\theta_0) \ln\left(\lambda^2\right) w_{a,b}(\lambda)\, d\lambda - \int_{\mathbb{R}^1} \psi(\lambda,\theta_0) \ln\left(1+\lambda^2\right) \ln\left(\lambda^2\right) w_{a,b}(\lambda)\, d\lambda].$$

The proofs of Theorems 1 and 2 are given in Section 4. The next section contains a generalization of the above results to random fields.

3. Estimation of parameters of random fields

BI. Let $Y(t), t \in [0,T]^n$, be an observation of a real-valued measurable stationary Gaussian random field $Y(t)$, $t \in \mathbb{R}^n$, with zero mean and spectral density $f(\lambda;\theta)$, $\lambda \in \mathbb{R}^n$, $\theta \in \Theta \subset \mathbb{R}^m$, where Θ is a compact set. Assume that the true value of the parameter vector $\theta_0 \in int\ \Theta$.

BII. For $\theta_1 \neq \theta_2$, $f(\lambda;\theta_1) \not\equiv f(\lambda;\theta_2)$ almost everywhere in $\mathbb{R}^n$ with respect to Lebesgue measure.

BIII. There exists a nonnegative function $w(\lambda), \lambda \in \mathbb{R}^n$, such that
1) $w(\lambda)$ is symmetric about $\lambda = 0 : w(\lambda) = w(-\lambda)$;
2) $w(\lambda) f(\lambda;\theta) \in L_1(\mathbb{R}^n)\ \forall\theta \in \Theta$.
Under the condition BIII, we set

$$\sigma^2(\theta) = \int_{\mathbb{R}^n} f(\lambda;\theta) w(\lambda) d\lambda \tag{3.1}$$

and consider the factorization

$$f(\lambda;\theta) = \sigma^2(\theta) \psi(\lambda;\theta), \quad \lambda \in \mathbb{R}^n, \theta \in \Theta. \tag{3.2}$$

For the function $\psi(\lambda,\theta), \lambda \in \mathbb{R}^n$, $\theta \in \Theta$, we have

$$\int_{\mathbb{R}^n} \psi(\lambda;\theta) w(\lambda) d\lambda = 1. \tag{3.3}$$

We additionally assume
BIV. The derivatives $\nabla_\theta \psi(\lambda;\theta)$ exist and

$$\nabla_\theta \int_{\mathbb{R}^n} \psi(\lambda;\theta) w(\lambda) d\lambda = \int_{\mathbb{R}^n} \nabla_\theta \psi(\lambda;\theta) w(\lambda) d\lambda = 0, \tag{3.4}$$

that is, we can differentiate under the integral sign in (3.3).
We define the periodogram

$$I_T(\lambda) = \frac{1}{(2\pi T)^n} \left| \int_{[0,T]^n} e^{-i(\lambda,t)} Y(t) dt \right|^2 \tag{3.5}$$

and consider the following contrast field:

$$U_T(\theta) = -\int_{\mathbb{R}^n} I_T(\lambda) w(\lambda) \log \psi(\lambda;\theta) d\lambda, \quad \theta \in \Theta. \tag{3.6}$$

We also define the functions

$$K(\theta_0;\theta) = \int_{\mathbb{R}^n} f(\lambda;\theta_0) w(\lambda) \log \frac{\psi(\lambda;\theta_0)}{\psi(\lambda;\theta)} d\lambda, \quad \theta_0, \theta \in \Theta \tag{3.7}$$

and

$$U(\theta) = -\int_{\mathbb{R}^n} f(\lambda;\theta_0) w(\lambda) \log \psi(\lambda;\theta) d\lambda. \tag{3.8}$$

BV. The function $w(\lambda), \lambda \in \mathbb{R}^n$, satisfies

$$f(\lambda;\theta_0) w(\lambda) \log \psi(\lambda;\theta) \in L_1(\mathbb{R}^n) \cap L_2(\mathbb{R}^n), \quad \forall\theta \in \Theta.$$

BVI. There exists a function $v(\lambda), \lambda \in \mathbb{R}^n$, such that
(i) the function $h(\lambda;\theta) = v(\lambda) \log \psi(\lambda;\theta)$ is uniformly continuous in $\mathbb{R}^n \times \Theta$;
(ii) $f(\lambda;\theta_0) \frac{w(\lambda)}{v(\lambda)} \in L_1(\mathbb{R}^n) \cap L_2(\mathbb{R}^n)$.

Theorem 3. *Let the conditions BI - BIII and BV -BVI be satisfied. Then the function $K(\theta_0;\theta)$ defined by (3.7) is the contrast function for the contrast field $U_T(\theta)$ defined by (3.6). Moreover the minimum contrast estimator $\widehat{\theta}_T$ defined as*

$$\widehat{\theta}_T = \arg\min_{\theta \in \Theta} U_T(\theta) \tag{3.9}$$

is a consistent estimator of the parameter vector θ*, that is,* $\widehat{\theta}_T \longrightarrow \theta_0$ *in* P_0*-probability as* $T \longrightarrow \infty$*, and the estimator*

$$\widehat{\sigma}_T^2 = \int_{\mathbb{R}^n} I_T(\lambda)\, w(\lambda)\, d\lambda$$

is a consistent estimator of the parameter $\sigma^2(\theta)$*, that is,* $\widehat{\sigma}_T^2 \longrightarrow \sigma^2(\theta_0)$ *in* P_0*-probability as* $T \longrightarrow \infty$.

The proof will be given in Section 4.

Remark 4. *The condition BVI is a technical one in comparison with the other conditions, which are more essential.*

Theorem 3 gives us the consistency (in a weak sense) of the estimator (3.9). However, due to the biasedness of (3.5) in the multidimensional case, we cannot obtain $T^{n/2}$-consistency and asymptotic normality of the estimator (3.9) constructed with the use of the periodogram (3.5). This leads to the consideration of the "unbiased" periodogram instead of (3.5). Note that another possibility to avoid the bias problem is to use data tapering.

Consider an unbiased estimator for the correlation function $B(t)$, $t \in \mathbb{R}^n$, of a random field $Y(t)$, $t \in \mathbb{R}^n$, satisfying condition BI (see, for example, Ivanov and Leonenko (1989)), namely,

$$\widehat{B}_T(t) = \prod_{j=1}^{n} (T - t_j)^{-1} \int_{D_T} Y(s) Y(t+s) ds,$$

where $D_T = \{s \in \mathbb{R}^n : s, s+t \in [0,T]^n\}$. Introduce the unbiased periodogram

$$I_T^*(\lambda) = \frac{1}{(2\pi)^n} \int_{[0,T]^n} \widehat{B}_T(t) e^{-i(\lambda,t)} dt, \ \ \lambda \in \mathbb{R}^n. \tag{3.10}$$

Denote

$$U_T^*(\theta) = -\int_{\mathbb{R}^n} I_T^*(\lambda)\, w(\lambda) \log \psi(\lambda;\theta)\, d\lambda, \quad \theta \in \Theta, \tag{3.11}$$

$$\widehat{\theta}_T^* = \arg\min_{\theta \in \Theta} U_T^*(\theta), \tag{3.12}$$

and

$$\widehat{\sigma}_T^{2*} = \int_{\mathbb{R}^n} I_T^*(\lambda)\, w(\lambda)\, d\lambda.$$

Theorem 3′. *Theorem 3 holds true for the contrast field* $U_T^*(\theta)$ *and estimators* $\widehat{\theta}_T^*$ *and* $\widehat{\sigma}_T^{2*}$*, constructed with the use of the unbiased periodogram (3.10).*

To formulate the result on the asymptotic distribution of the minimum contrast estimator (3.12) we need some more conditions:

BVII. The function $\psi(\lambda;\theta)$ is twice differentiable in a neighborhood of the point θ_0 and

1) $f(\lambda;\theta_0)\, w(\lambda) \frac{\partial^2}{\partial\theta_i \partial\theta_j} \log \psi(\lambda,\theta) \in L_1(\mathbb{R}^n) \cap L_2(\mathbb{R}^n)$, $i,j = 1,...,m$, $\theta \in \Theta$;

2) $f(\lambda;\theta_0)\, w(\lambda) \frac{\partial}{\partial\theta_i} \log \psi(\lambda;\theta) \in L_k(\mathbb{R}^n)$ for all $k \geq 1$, $i = 1,...,m$, $\theta \in \Theta$.

BVIII. The matrices

$$S(\theta) = (s_{ij}(\theta))_{i,j=1,\dots,m}$$

and

$$A(\theta) = (a_{ij}(\theta))_{i,j=1,\dots,m}$$

are positive definite, where

$$s_{ij}(\theta) = \int_{\mathbb{R}^n} f(\lambda;\theta)\, w(\lambda) \frac{\partial^2}{\partial\theta_i \partial\theta_j} \log\psi(\lambda;\theta)\, d\lambda$$

$$= \sigma^2(\theta) \int_{\mathbb{R}^n} w(\lambda) \left[\frac{\partial^2}{\partial\theta_i\partial\theta_j}\psi(\lambda,\theta) - \frac{1}{\psi(\lambda,\theta)} \frac{\partial}{\partial\theta_i}\psi(\lambda,\theta) \frac{\partial}{\partial\theta_j}\psi(\lambda,\theta) \right] d\lambda,$$

$$a_{ij}(\theta) = 2(2\pi)^n \int_{\mathbb{R}^n} f^2(\lambda;\theta)\, w^2(\lambda) \frac{\partial}{\partial\theta_i}\log\psi(\lambda;\theta) \frac{\partial}{\partial\theta_j}\log\psi(\lambda;\theta)\, d\lambda$$

$$= 2(2\pi)^n \left(\sigma^2(\theta)\right)^2 \int_{\mathbb{R}^n} w^2(\lambda) \frac{\partial}{\partial\theta_i}\psi(\lambda;\theta) \frac{\partial}{\partial\theta_j}\psi(\lambda;\theta)\, d\lambda,$$

$$i,j = 1,\dots,m.$$

BIX. The spectral density $f(\lambda;\theta)$, the weight function $w(\lambda)$ and the functions $\frac{\partial}{\partial\theta_i}\log\psi(\lambda;\theta)$, $i = 1,\dots m$, are such that

$$T^{n/2} \int_{\mathbb{R}^n} (EI_T^*(\lambda) - f(\lambda;\theta_0)) w(\lambda) \frac{\partial}{\partial\theta_i} \log\psi(\lambda;\theta)\, d\lambda \to 0$$

as $T \to \infty$.

Theorem 4. *Let the conditions BI - BIX be satisfied. Then as* $T \longrightarrow \infty$

$$T^{n/2}\left(\widehat{\theta}_T^* - \theta_0\right) \xrightarrow{\mathcal{D}} \mathcal{N}_m\left(0, S^{-1}(\theta_0)\, A(\theta_0)\, S^{-1}(\theta_0)\right), \tag{3.13}$$

where the matrices $S(\theta)$ *and* $A(\theta)$ *are defined in condition BVIII and* $\mathcal{N}_m(\cdot,\cdot)$ *denotes the* m*-dimensional Gaussian law.*

The proof will be given in Section 4.

Remark 5. *If we drop the condition BIX in the formulation of Theorem 4, then the asymptotic normality stated in this theorem will hold for the variables* $T^{n/2}\left(\widehat{\theta}_T - E\widehat{\theta}_T\right)$ *and* $T^{n/2}\left(\widehat{\theta}_T^* - E\widehat{\theta}_T^*\right)$.

Remark 6. *For the case of processes* $(n = 1)$*, the condition BIX holds under any assumptions on* $f(\lambda;\theta)$, $w(\lambda)$ *and* $\frac{\partial}{\partial\theta_i}\log\psi(\lambda;\theta)$, $i = 1,\dots,m$, *which guarantee*

$$\int_{\mathbb{R}^1} |f(\lambda+h;\theta_0) - f(\lambda;\theta_0)|\, |w(\lambda) \frac{\partial}{\partial\theta_i}\log\psi(\lambda;\theta)|\, d\lambda \le C|h|^\alpha$$

with $\alpha > 1/2$ *and* $C > 0$ *being a constant. For example, we can apply the minimum contrast estimation technique based on the contrast process (3.11) to the Gaussian random process* $X_1(t,x)$, $t > 0$, $x \in \mathbb{R}^1$, *which appears as an approximation of the rescaled solution of the heat equation with singular initial data. For more details, we refer the* *reader to Anh and Leonenko* [5]*. We just note here that this Gaussian process is stationary in x, has zero mean and, for fixed $t > 0$, its spectral density is of the form*

$$f(\lambda, \varkappa) = const\,\frac{e^{-2\mu t\lambda^2}}{|\lambda|^{1-\varkappa}}, \quad \lambda \in \mathbb{R}^1.$$

In this case, we may choose $w(\lambda) = |\lambda|^a$, $a > 2$.

4. Proofs

Proof of Theorem 1.

Let us show that as $T \longrightarrow \infty$

$$\begin{aligned} U_T(\theta) &= -\int_{-\infty}^{\infty} I_T(\lambda) \log \psi(\lambda;\theta)\, w_{ab}(\lambda)\, d\lambda \\ &\longrightarrow U(\theta) = -\int_{-\infty}^{\infty} f(\lambda;\theta_0) \log \psi(\lambda;\theta)\, w_{ab}(\lambda)\, d\lambda \end{aligned} \tag{4.1}$$

in P_0-probability. Denote

$$\varphi(\lambda) = \varphi(\lambda;\theta) = \log \psi(\lambda;\theta)\, w_{ab}(\lambda) = \log \psi(\lambda;\theta)\, \frac{|\lambda|^{2b}}{\left(1+\lambda^2\right)^a}.$$

To prove (4.1) we show that

$$\int_{-\infty}^{\infty} \left(EI_T(\lambda) - f(\lambda;\theta_0)\right) \varphi(\lambda;\theta)\, d\lambda \longrightarrow 0 \tag{4.2}$$

and

$$\int_{-\infty}^{\infty} \left(I_T(\lambda) - E\left(I_T(\lambda)\right)\right) \varphi(\lambda;\theta)\, d\lambda \longrightarrow 0 \tag{4.3}$$

in probability. Consider

$$EI_T(\lambda) = E\left\{\frac{1}{2\pi T}\int_0^T \int_0^T Y(s)\, Y(t)\, e^{i(s-t)\lambda} ds dt\right\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(w;\theta_0)\, \frac{\sin^2 \frac{T}{2}(\lambda - w)}{T\left(\frac{\lambda-w}{2}\right)^2} dw.$$

Then, as $T \longrightarrow \infty$ we have

$$E\int_{-\infty}^{\infty} I_T(\lambda)\, \varphi(\lambda;\theta)\, d\lambda \longrightarrow \int_{-\infty}^{\infty} f(\lambda;\theta_0)\, \varphi(\lambda;\theta)\, d\lambda$$

provided that

$$\int_{-\infty}^{\infty} f(\lambda;\theta_0)\, \varphi(\lambda;\theta)\, d\lambda < \infty. \tag{4.4}$$

Consider the integral (4.4):

$$\int_{\mathbb{R}^1} f(\lambda;\theta_0)\, \varphi(\lambda;\theta)\, d\lambda = \int_{\mathbb{R}^1} f(\lambda;\theta_0) \log \psi(\lambda;\theta)\, \frac{|\lambda|^{2b}}{\left(1+\lambda^2\right)^a} d\lambda$$

$$\begin{aligned} &= \int_{\mathbb{R}^1} \frac{\eta_0}{|\lambda|^{2\beta_0}\left(1+\lambda^2\right)^{\alpha_0}} \left[\log \frac{\eta}{|\lambda|^{2\beta}\left(1+\lambda^2\right)^{\alpha}} - \log \sigma^2(\theta)\right] \frac{|\lambda|^{2b}}{\left(1+\lambda^2\right)^a} d\lambda \\ &= \int_{\mathbb{R}^1} \frac{\eta_0}{|\lambda|^{2\beta_0 - 2b}\left(1+\lambda^2\right)^{\alpha_0 + a}} \left[-2\beta \log|\lambda| - \alpha \log\left(1+\lambda^2\right) + \log \eta - \log \sigma^2(\theta)\right] d\lambda. \end{aligned}$$

As $\lambda \longrightarrow 0$, the integrand tends to zero in view of $\frac{-\log|\lambda|}{|\lambda|^{2\beta_0-2b}} \longrightarrow 0$ as long as $b > \beta_0$. As $\lambda \longrightarrow \infty$, we can write

$$\frac{\log|\lambda|}{|\lambda|^{2\beta_0-2b}\left(1+\lambda^2\right)^{\alpha_0+a}} = O\left(\frac{\lambda}{\lambda^{2(\alpha_0+a+\beta_0-b)}}\right).$$

Therefore, under the conditions $b > 1/2$, $a > b+1/2$, we will have $b > \beta_0$ and $a > b+1-\beta_0-\alpha_0$, which provides the convergence of the integral (4.4).

In order to prove (4.3), we show that

$$E\left\{\int_{-\infty}^{\infty}\left(I_T(\lambda)-E\left(I_T(\lambda)\right)\right)\varphi(\lambda;\theta)\,d\lambda\right\}^2 \longrightarrow 0. \tag{4.5}$$

In what follows, we will omit the dependence on parameter θ in functions $\varphi(\lambda;\theta)$ and $f(\lambda;\theta)$ and we will write $f_0(\lambda)$ for $f(\lambda;\theta_0)$. We have

$$E\left\{\int_{-\infty}^{\infty}\left(I_T(\lambda)-E\left(I_T(\lambda)\right)\right)\varphi(\lambda;\theta)\,d\lambda\right\}^2$$

$$\begin{aligned}
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[E\left(I_T(\lambda)\,I_T(\lambda')\right)-E\left(I_T(\lambda)\right)E(I_T(\lambda'))\right]\varphi(\lambda)\,\varphi(\lambda')\,d\lambda d\lambda' \\
&= \frac{1}{4\pi^2T^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{[0,T]^4}\left[E\left(Y(t)\,Y(s)\,Y(t')\,Y(s')\right)-E\left(Y(t)\,Y(s)\right)E(Y(t')\,Y(s'))\right] \\
&\times e^{-i\lambda(t-s)-i\lambda'(t'-s')}dtdsdt'ds'\varphi(\lambda)\,\varphi(\lambda')\,d\lambda d\lambda'. \qquad (4.6)
\end{aligned}$$

Since $Y(t), t\in\mathbb{R}^1$, is a zero-mean Gaussian process, we have

$$\begin{aligned}
& E\left(Y(t)\,Y(s)\,Y(t')\,Y(s')\right)-E\left(Y(t)\,Y(s)\right)E\left(Y(t')\,Y(s')\right) \\
= \;& E\left(Y(t)\,Y(t')\right)E\left(Y(s)\,Y(s')\right)+E\left(Y(t)\,Y(s')\right)E\left(Y(s)\,Y(t')\right) \qquad (4.7)
\end{aligned}$$

(see, for example, Hannan [26]). Consider

$$\frac{1}{4\pi^2T^2}\int_{[0,T]^4}E\left(Y(t)\,Y(t')\right)E\left(Y(s)\,Y(s')\right)e^{-i\lambda(t-s)-i\lambda'(t'-s')}dtdsdt'ds'$$

$$\begin{aligned}
&= \frac{1}{4\pi^2T^2}\int_{[0,T]^4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f_0(x)\,f_0(y)\,e^{i(t-t')x+i(s-s')y}dxdy \\
&\quad\times e^{-i\lambda(t-s)-i\lambda'(t'-s')}dtdsdt'ds' \\
&= \frac{2\pi}{T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f_0(x)\,f_0(y)\,\frac{1}{(2\pi)^3\,T}\int_{[0,T]^4}e^{it(x-\lambda)+is(y+\lambda)} \\
&\quad\times e^{it'(-x-\lambda')+is'(-y+\lambda')}dtdsdt'ds'dxdy \\
&= \frac{2\pi}{T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f_0(x)\,f_0(y)\,\Phi_T^4\left(x-\lambda, y+\lambda, -x-\lambda'\right)dxdy, \qquad (4.8)
\end{aligned}$$

where we have denoted by $\Phi_T^4(u_1,u_2,u_3)$ the following kernel:

$$\Phi_T^4(u_1,u_2,u_3) = \Phi_T^4(u_1,u_2,u_3,u_4) = \frac{1}{(2\pi)^3\,T}\int_{[0,T]^4}e^{i(t_1u_1+t_2u_2+t_3u_3+t_4u_4)}dt_1...dt_4 \tag{4.9}$$

with $u_4 = -(u_1+u_2+u_3)$ (see Appendix A for more details). Similarly,

$$\frac{1}{4\pi^2T^2}\int_{[0,T]^4} E\left(Y(t)Y(s')\right)E\left(Y(s)Y(t')\right)e^{-i\lambda(t-s)-i\lambda'(t'-s')}dtdsdt'ds'$$

$$= \frac{2\pi}{T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_0(x)\,f_0(y)\,\Phi_T^4(x-\lambda, y+\lambda, -x+\lambda')\,dxdy. \tag{4.10}$$

In view of (4.7), (4.8) and (4.10), we arrive at the following expression for (4.6):

$$E\left\{\int_{-\infty}^{\infty}\left(I_T(\lambda) - E(I_T(\lambda))\right)\varphi(\lambda)\,d\lambda\right\}^2$$

$$\begin{aligned}
&= \frac{2\pi}{T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi(\lambda)\,\varphi(\lambda')\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_0(x)\,f_0(y)\\
&\quad\times\left[\Phi_T^4(x-\lambda, y+\lambda, -x-\lambda') + \Phi_T^4(x-\lambda, y+\lambda, -x+\lambda')\right]dxdyd\lambda d\lambda'\\
&= \frac{2\pi}{T}\int_{\mathbb{R}^3}\Phi_T^4(u_1,u_2,u_3)\int_{\mathbb{R}^1}[\varphi(\lambda)\,\varphi(-u_3-u_1-\lambda)\,f_0(u_1+\lambda)\,f_0(u_2-\lambda)\\
&\quad+\varphi(\lambda)\,\varphi(u_1+u_3+\lambda)\,f_0(u_1+\lambda)\,f_0(u_2-\lambda)]d\lambda du_1du_2du_3\\
&= \frac{2\pi}{T}\int_{\mathbb{R}^3}\Phi_T^4(u_1,u_2,u_3)\,G(u_1,u_2,u_3)\,du_1du_2du_3,
\end{aligned} \tag{4.11}$$

where

$$G(u_1,u_2,u_3) = 2\int_{\mathbb{R}^1}\varphi(\lambda)\,\varphi(u_1+u_3+\lambda)\,f_0(u_1+\lambda)\,f_0(u_2-\lambda)\,d\lambda. \tag{4.12}$$

Using the properties of the kernel $\Phi_T^4(u_1,u_2,u_3)$ (see Appendix A) and properties of the function $G(u_1,u_2,u_3)$, and in particular, the fact that, under the assumption $b > 1/2$, $a > b + 1/2$,

$$G(0,0,0) = 2\int_{-\infty}^{\infty}\varphi^2(\lambda)\,f_0^2(\lambda)\,d\lambda < \infty,$$

we come to the conclusion that

$$E\left\{\int_{-\infty}^{\infty}\left(I_T(\lambda) - E(I_T(\lambda))\right)\varphi(\lambda)\,d\lambda\right\}^2 = O\left(\frac{1}{T}\right),$$

which completes the proof of (4.3), and so the convergence (4.1) takes place.

From (4.1) it follows that in P_0-probability as $T \longrightarrow \infty$

$$U_T(\theta) - U_T(\theta_0) \longrightarrow U(\theta) - U(\theta_0) = K(\theta_0;\theta),$$

where $K(\theta_0;\theta)$ is given by (2.11).

Let us now prove that $K(\theta_0;\theta) \geq 0$ with the equality holds if and only if $\theta = \theta_0$. Indeed, using Jensen's inequality and relations (2.3) - (2.5) we have

$$\begin{aligned}
-K(\theta_0;\theta) &= \int_{\mathbb{R}^1} f(\lambda;\theta_0)\,w_{ab}(\lambda)\log\frac{\psi(\lambda;\theta)}{\psi(\lambda;\theta_0)}d\lambda\\
&= \sigma^2(\theta_0)\int_{\mathbb{R}^1}\psi(\lambda;\theta_0)\,w_{ab}(\lambda)\log\frac{\psi(\lambda;\theta)}{\psi(\lambda;\theta_0)}d\lambda\\
&\leq \sigma^2(\theta_0)\log\int_{\mathbb{R}^1}\psi(\lambda;\theta)\,w_{ab}(\lambda)\,d\lambda = 0,
\end{aligned}$$

that is, $K(\theta_0;\theta) \geq 0$. Moreover, $K(\theta_0;\theta) > 0$ if $\psi(\lambda;\theta_0) \not\equiv \psi(\lambda;\theta)$ for $\theta \neq \theta_0$ almost everywhere with respect to Lebesgue measure. This completes the proof of the first part of the theorem.

Now, for the consistency of the estimator $\widehat{\theta}_T$ given by (2.12), we just need to prove that the convergence (4.1) holds uniformly in $\theta \in \Theta$ (see, for example, Guyon [25], Theorem 3.4.1). Consider for arbitrary $\theta_1, \theta_2 \in \Theta$ with $\theta_i = (\alpha_i, \beta_i)$, $i = 1, 2$

$$
\begin{aligned}
|U_T(\theta_1) - U_T(\theta_2)| &\leq \int_{\mathbb{R}^1} \left| \log \frac{\psi(\lambda;\theta_1)}{\psi(\lambda;\theta_2)} w_{ab}(\lambda) \right| I_T(\lambda)\, d\lambda \\
&\leq 2|\beta_2 - \beta_1| \int_{\mathbb{R}^1} |\log|\lambda|| w_{ab}(\lambda)\, I_T(\lambda)\, d\lambda \\
&\quad + |\alpha_2 - \alpha_1| \int_{\mathbb{R}^1} \log(1+\lambda^2) w_{ab}(\lambda)\, I_T(\lambda)\, d\lambda \\
&\quad + |\log \widetilde{B}(\alpha_2,\beta_2) - \log \widetilde{B}(\alpha_1,\beta_1)| \int_{\mathbb{R}^1} w_{ab}(\lambda)\, I_T(\lambda)\, d\lambda,
\end{aligned} \tag{4.13}
$$

where we have denoted $\widetilde{B}(\alpha,\beta) = B(\frac{1}{2} - \beta + b, \alpha + a - \frac{1}{2} + \beta - b)$.

In view of (4.13) and Theorems 21.9 and 21.10 of Davidson [17], in order to prove the uniform convergence in (4.1), it is sufficient to show that as $T \longrightarrow \infty$

$$
J_T^{(i)} = \int_{\mathbb{R}^1} g_i(\lambda) w_{ab}(\lambda)\, I_T(\lambda)\, d\lambda = O_p(1), \quad i = 1,2,3, \tag{4.14}
$$

where $g_1(\lambda) = |\log|\lambda||$, $g_2(\lambda) = \log(1+\lambda^2)$, $g_3(\lambda) \equiv 1$.

The relations (4.14) will follow if

$$
\int_{\mathbb{R}^1} g_i(\lambda) w_{ab}(\lambda)\, f_0(\lambda) d\lambda < \infty, \quad i = 1,2,3, \tag{4.15}
$$

$$
EJ_T^{(i)} = E \int_{\mathbb{R}^1} g_i(\lambda) w_{ab}(\lambda)\, I_T(\lambda)\, d\lambda \to \int_{\mathbb{R}^1} g_i(\lambda) w_{ab}(\lambda)\, f_0(\lambda) d\lambda, \quad i = 1,2,3, \tag{4.16}
$$

and

$$
J_T^{(i)} - EJ_T^{(i)} \longrightarrow 0 \tag{4.17}
$$

in probability. The last relation (4.17) will hold if

$$
E(J_T^{(i)} - EJ_T^{(i)})^2 \longrightarrow 0. \tag{4.18}
$$

The proof of the relations (4.15), (4.16) and (4.18) can be done in the same manner as that of (4.4), (4.2) and (4.5) respectively, and, moreover, if we reconsider the above mentioned proofs, we will find that the results (4.15), (4.16) and (4.18) can be simply extracted from the proofs for (4.4), (4.2) and (4.5). Thus, the proof of the consistency of the minimum contrast estimator $\widehat{\theta}_T$ is completed.

Remark 7. *The proof of the relation (4.1) reveals that a more general result is true, namely, the following*

Lemma 1. *Let $Y(t), t \in \mathbb{R}^1$, be a stationary Gaussian process with spectral density function $f(\lambda)$, $\lambda \in \mathbb{R}^1$, and $\varphi(\lambda)$, $\lambda \in \mathbb{R}^1$, be a nonrandom function such that*

1) $\varphi(\lambda)$ is symmetric about $\lambda = 0 : \varphi(\lambda) = \varphi(-\lambda)$;

2) $f(\lambda)\varphi(\lambda) \in L_1(\mathbb{R}^1) \cap L_2(\mathbb{R}^1)$.

Let $I_T(\lambda)$ *be the periodogram based on* $Y(t), t \in [0,T] : I_T(\lambda) = \frac{1}{2\pi T}\left|\int_0^T e^{-i\lambda t} Y(t)\,dt\right|^2$. *Then, as* $T \longrightarrow \infty$

$$\int_{\mathbb{R}^1} I_T(\lambda)\varphi(\lambda)\,d\lambda \longrightarrow \int_{\mathbb{R}^1} f(\lambda)\varphi(\lambda)\,d\lambda$$

in probability.

Proof of Theorem 2.

From Taylor's formula we have

$$\nabla_\theta U_T\left(\widehat{\theta}_T\right) = \nabla_\theta U_T(\theta_0) + \nabla_\theta \nabla'_\theta U_T(\theta_T^*)\left(\widehat{\theta}_T - \theta_0\right),$$

where $|\theta_T^* - \theta_0| < \left|\widehat{\theta}_T - \theta_0\right|$;

$$\nabla_\theta U_T(\theta) = -\int_{\mathbb{R}^1} I_T(\lambda)\, w_{ab}(\lambda)\, \nabla_\theta \log \psi(\lambda;\theta)\,d\lambda,$$

$$\nabla_\theta \nabla'_\theta U_T(\theta) = \int_{\mathbb{R}^1} I_T(\lambda)\, w_{ab}(\lambda) \left(\frac{\partial^2}{\partial\theta_i \partial\theta_j} \log \psi(\lambda;\theta)\right)_{i,j=1,2} d\lambda.$$

It follows from the definition of minimum contrast estimators that for T sufficiently large

$$\nabla_\theta U_T(\theta_0) = -\nabla_\theta \nabla'_\theta U_T(\theta_T^*)\left(\widehat{\theta}_T - \theta_0\right). \tag{4.19}$$

If we can show that as $T \longrightarrow \infty$

$$\nabla_\theta \nabla'_\theta U_T(\theta_T^*) \longrightarrow S(\theta_0) \tag{4.20}$$

in P_0-probability, where the matrix $S(\theta_0)$ is given by (2.17), and

$$\sqrt{T}\nabla_\theta U_T(\theta_0) \xrightarrow{D} \mathcal{N}_2(0, A(\theta_0)), \tag{4.21}$$

where the matrix $A(\theta_0)$ is given by (2.18), then

$$\sqrt{T} S(\theta_0)\left(\widehat{\theta}_T - \theta_0\right) \xrightarrow{D} \mathcal{N}_2(0, A(\theta_0)),$$

and, by Slutsky's arguments, the relation (2.19) is a consequence of (4.19), (4.20) and (4.21). So we need to prove (4.20) and (4.21). Denote

$$Q(\lambda) = (q_{ij}(\lambda))_{i,j=1,2} = \left(w_{ab}(\lambda)\frac{\partial^2}{\partial\theta_i \partial\theta_j} \log \psi(\lambda;\theta)\right)_{i,j=1,2}.$$

The functions $q_{ij}(\lambda), i,j = 1,2$, satisfy the conditions of Lemma 4.1, that is,

$$f(\lambda,\theta_0)\, q_{ij}(\lambda) \in L_1\left(\mathbb{R}^1\right) \tag{4.22}$$

and

$$f(\lambda,\theta_0)\, q_{ij}(\lambda) \in L_2\left(\mathbb{R}^1\right). \tag{4.23}$$

Indeed, we have the following expressions for the second order derivatives of the function $\psi(\lambda;\theta)$:

$$\frac{\partial^2\psi}{\partial\alpha^2} = \psi(\lambda;\theta)\,[\left(\ln\left(1+\lambda^2\right)\right)^2 - 2\ln\left(1+\lambda^2\right)\int_{\mathbb{R}^1}\psi(\lambda;\theta)\ln\left(1+\lambda^2\right)w_{ab}(\lambda)\,d\lambda$$

$$-\int_{\mathbb{R}^1}\psi(\lambda;\theta)\left\{\ln\left(1+\lambda^2\right)\right\}^2 w_{ab}(\lambda)\,d\lambda + 2\left\{\int_{\mathbb{R}^1}\psi(\lambda;\theta)\ln\left(1+\lambda^2\right)w_{ab}(\lambda)\,d\lambda\right\}^2];$$

$$\frac{\partial^2\psi}{\partial\beta^2} = \psi(\lambda;\theta)\,[\left(\ln\lambda^2\right)^2 - 2\ln\lambda^2\int_{\mathbb{R}^1}\psi(\lambda;\theta)\ln\lambda^2 w_{ab}(\lambda)\,d\lambda$$

$$-\int_{\mathbb{R}^1}\psi(\lambda;\theta)\left(\ln\lambda^2\right)^2 w_{ab}(\lambda)\,d\lambda + 2\left\{\int_{\mathbb{R}^1}\psi(\lambda;\theta)\ln\lambda^2 w_{ab}(\lambda)\,d\lambda\right\}^2];$$

$$\frac{\partial^2\psi}{\partial\alpha\partial\beta} = \psi(\lambda;\theta)\,[\ln\lambda^2\ln\left(1+\lambda^2\right) - \ln\lambda^2\int_{\mathbb{R}^1}\psi(\lambda;\theta)\ln\left(1+\lambda^2\right)w_{ab}(\lambda)\,d\lambda$$

$$-\ln\left(1+\lambda^2\right)\int_{\mathbb{R}^1}\psi(\lambda;\theta)\ln\lambda^2 w_{ab}(\lambda)\,d\lambda - \int_{\mathbb{R}^1}\psi(\lambda;\theta)\ln\left(1+\lambda^2\right)\ln\lambda^2 w_{ab}(\lambda)\,d\lambda$$
$$+2\int_{\mathbb{R}^1}\psi(\lambda;\theta)\ln\left(1+\lambda^2\right)w_{ab}(\lambda)\,d\lambda\int_{\mathbb{R}^1}\psi(\lambda;\theta)\ln\lambda^2 w_{ab}(\lambda)\,d\lambda].$$

Our choice of parameters a and b, which is $b>1, a>b+5/4$, guarantees that

$$f(\lambda;\theta)\,\psi(\lambda;\theta)\,w_{ab}(\lambda)\,g(\lambda)\quad\in L_1\left(\mathbb{R}^1\right)\cap L_2\left(\mathbb{R}^1\right),$$

with $g(\lambda)$ taken to be equal respectively to the following functions:

$$\left(\ln\left(1+\lambda^2\right)\right)^2, \ln\left(1+\lambda^2\right), \left(\ln\lambda^2\right)^2, \ln\lambda^2, \ln\left(1+\lambda^2\right)\ln\lambda^2.$$

This means that the assertions (4.22) and (4.23) hold in view of the form of the second derivatives of the function $\psi(\lambda;\theta)$ given above. Therefore, in conjunction with Lemma 1, we have

$$\int_{\mathbb{R}^1} I_T(\lambda)\,Q(\lambda)\,d\lambda \longrightarrow \int_{\mathbb{R}^1} f(\lambda;\theta_0)\,Q(\lambda)\,d\lambda$$

in P_0-probability, that is, (4.20) holds.

We will now prove (4.21). The elements of the matrix $A(\theta_0)=\{a_{ij}(\theta_0)\}_{i,j=1,2}$ can be written in the form

$$\begin{aligned}
a_{11}(\theta_0) &= 4\pi\int_{\mathbb{R}^1} f^2(\lambda;\theta_0)\,\varphi_\alpha^2(\lambda;\theta_0)\,d\lambda;\\
a_{22}(\theta_0) &= 4\pi\int_{\mathbb{R}^1} f^2(\lambda;\theta_0)\,\varphi_\beta^2(\lambda;\theta_0)\,d\lambda;\\
a_{12}(\theta_0) &= a_{21}(\theta_0) = 4\pi\int_{\mathbb{R}^1} f^2(\lambda;\theta_0)\,\varphi_\alpha(\lambda;\theta_0)\,\varphi_\beta(\lambda;\theta_0)\,d\lambda,
\end{aligned}$$

where

$$\begin{aligned}\varphi_{\alpha}(\lambda;\theta_0) &= w_{ab}(\lambda)\frac{\partial}{\partial\alpha}\log\psi(\lambda;\theta_0),\\ \varphi_{\beta}(\lambda;\theta_0) &= w_{ab}(\lambda)\frac{\partial}{\partial\beta}\log\psi(\lambda;\theta_0).\end{aligned}$$

Consider the vector

$$\left(\frac{\partial}{\partial\alpha}U_T(\theta_0),\frac{\partial}{\partial\beta}U_T(\theta_0)\right)' = \left(-\int_{\mathbb{R}^1} I_T(\lambda)\varphi_{\alpha}(\lambda;\theta_0)\,d\lambda, -\int_{\mathbb{R}^1} I_T(\lambda)\varphi_{\beta}(\lambda;\theta_0)\,d\lambda\right)'.$$

Let c_α, c_β be fixed constants and consider the random variable

$$\begin{aligned}Y_T &= c_\alpha\frac{\partial}{\partial\alpha}U_T(\theta_0) + c_\beta\frac{\partial}{\partial\beta}U_T(\theta_0)\\ &= \int_{\mathbb{R}^1} I_T(\lambda)\left[c_\alpha\varphi_\alpha(\lambda;\theta_0) + c_\beta\varphi_\beta(\lambda;\theta_0)\right]d\lambda\\ &= \int_{\mathbb{R}^1} I_T(\lambda)\Psi(\lambda;\theta_0)\,d\lambda.\end{aligned}$$

We will show that $T^{1/2}Y_T$ tends in distribution as $T \longrightarrow \infty$ to a normal random variable with mean zero and variance s^2 given by

$$\begin{aligned}s^2 &= 4\pi\int_{\mathbb{R}^1} f^2(\lambda;\theta_0)\Psi^2(\lambda;\theta_0)\,d\lambda\\ &= 4\pi\int_{\mathbb{R}^1} f^2(\lambda;\theta_0)\left\{c_\alpha\varphi_\alpha(\lambda;\theta_0) + c_\beta\varphi_\beta(\lambda;\theta_0)\right\}^2 d\lambda\\ &= 4\pi\sum_{i,j=1,2}\sum c_i c_j\int_{\mathbb{R}^1} f^2(\lambda;\theta_0)\varphi_i(\lambda;\theta_0)\varphi_j(\lambda;\theta_0)\,d\lambda\\ &= \sum_{i,j=1,2}\sum c_i c_j a_{ij}(\theta_0).\end{aligned}$$

Firstly, we notice that in view of (2.8)

$$\int_{\mathbb{R}^1} f(\lambda;\theta_0)\frac{\partial}{\partial\theta_j}\log\psi(\lambda;\theta_0)\,w_{ab}(\lambda)\,d\lambda = 0, \quad j = 1,2.$$

On the other hand, analogous to (4.2), we can show that as $T \longrightarrow \infty$

$$\int_{\mathbb{R}^1}\left(EI_T(\lambda) - f(\lambda;\theta_0)\right)\Psi(\lambda)\,d\lambda \longrightarrow 0.$$

Moreover, from the results of Bentkus *et al.* [13] concerning the rate of convergence of the first moment of spectral estimates, we deduce that under the imposed conditions on the parameters a, b of the weight function $w_{ab}(\lambda)$ we have

$$E(T^{1/2}Y_T) \longrightarrow 0 \text{ as } T \longrightarrow \infty.$$

Analogous to our derivation of the formula (4.11), we find that the variance

$$\begin{aligned} VarT^{1/2}Y_T &= TE\left\{\int_{\mathbb{R}^1} I_T(\lambda)\Psi(\lambda)\,d\lambda - E\int_{\mathbb{R}^1} I_T(\lambda)\Psi(\lambda)\,d\lambda\right\}^2 \\ &= TE\left\{\int_{\mathbb{R}^1} (I_T(\lambda) - EI_T(\lambda))\Psi(\lambda)\,d\lambda\right\}^2 \\ &= 2\pi\int_{\mathbb{R}^3} \Phi_T^4(u_1,u_2,u_3)\,G_2(u_1,u_2,u_3)\,du_1du_2du_3, \end{aligned}$$

where now

$$\begin{aligned} G_2(u_1,u_2,u_3) &= \int_{\mathbb{R}^1} [\Psi(\lambda)\Psi(-u_3-u_1-\lambda)f(u_1+\lambda;\theta_0)f(u_2-\lambda;\theta_0)] \\ &\quad +\Psi(\lambda)\Psi(-u_3-u_2+\lambda)f(u_1+\lambda;\theta_0)f(u_2-\lambda;\theta_0)]d\lambda. \end{aligned}$$

Using the properties of the kernel $\Phi_T^4(u_1,u_2,u_3)$ (see Appendix A), we obtain that

$$VarT^{1/2}Y_T = 4\pi\int_{\mathbb{R}^1} \Psi^2(\lambda)\,f^2(\lambda;\theta_0)\,d\lambda. \tag{4.24}$$

Turning to the investigation of the cumulant of order $k \geq 3$ of the variable $T^{1/2}Y_T$, we have

$$\begin{aligned} C_K^T &= cum\left\{T^{1/2}Y_T,...,T^{1/2}Y_T\right\} \\ &= T^{k/2}\int_{\mathbb{R}^k}\prod_{i=1}^{k}\Psi(\lambda_i)\,cum\left\{I_T(\lambda_1),...I_T(\lambda_k)\right\}d\lambda_1...d\lambda_k. \end{aligned}$$

Consider

$$cum\left\{I_T(\lambda_1),...I_T(\lambda_k)\right\} = \frac{1}{(2\pi T)^k}$$

$$\times cum\left\{\int_0^T\int_0^T Y(t_1)Y(s_1)e^{-i(t_1-s_1)\lambda_1}dt_1ds_1,...,\int_0^T\int_0^T Y(t_k)Y(s_k)e^{-i(t_k-s_k)\lambda_k}dt_kds_k\right\}$$

$$= \frac{1}{(2\pi T)^k}\int_{[0,T]^{2k}} e^{-i\Sigma_{j=1}^k(t_j-s_j)\lambda_j}cum\left\{Y(t_1)Y(s_1),...,Y(t_k)Y(s_k)\right\}dt_1ds_1...dt_kds_k.$$

To calculate the cumulant appearing under the integral sign in the last expression, we use the diagram formula (see Appendix B) which gives

$$\begin{aligned} &cum\left\{Y(t_1)Y(s_1),...,Y(t_k)Y(s_k)\right\} \\ &= \sum_{\Gamma_{2,k}} cov(Y(t_1)Y(u_{j_2}))\,cov(Y(\overline{u}_{j_2}),Y(u_{j_3}))...cov(Y(\overline{u}_{j_k}),Y(s_1)), \end{aligned} \tag{4.25}$$

where the sum is taken over all complete closed diagrams $\Gamma_{2,k}$ with k levels and 2 vertices in each level, the number of such diagrams (and therefore the number of terms in the above sum) is $2^{k-1}(k-1)!$; and we have denoted here $u_j = t_j \vee s_j$ and $\overline{u}_j = \begin{cases} t_j, & \text{if } u_j = s_j \\ s_j, & \text{if } u_j = t_j \end{cases}$; and $(j_2,...,j_k)$ is a permutation of $(2,...,k)$. The simplicity of the expression for the cumulant (4.25) is due to the Gaussianity of the process $Y(t)$, for which all cumulants are zero except the second-order ones.

Continuing we arrive at the following expression for the cumulant:

$$\begin{aligned} & cum\left\{Y(t_1)Y(s_1),...,Y(t_k)Y(s_k)\right\} \\ = & \sum_{\Gamma_{2,k}}\int_{\mathbb{R}^k}\exp\{i(t_1-u_{j_2})\omega_1+i(\overline{u}_{j_2}-u_{j_3})\omega_2+...+i(\overline{u}_{j_k}-s_1)\omega_k\} \\ & \times f_0(\omega_1)...f_0(\omega_k)\,d\omega_1...d\omega_k, \end{aligned}$$

and therefore we can write

$$\begin{aligned} C_k^T = & T^{k/2}\sum_{\Gamma_{2,k}}\frac{1}{(2\pi T)^k}\int_{\mathbb{R}^k}\prod_{i=1}^k\Psi(\lambda_i)\int_{\mathbb{R}^k}\int_{[0,T]^{2k}}\exp\{-i\Sigma_{j=1}^k(t_j-s_j)\lambda_j\} \\ & \times\exp\{i(t_1-u_{j_2})\omega_1+i(\overline{u}_{j_2}-u_{j_3})\omega_2+...+i(\overline{u}_{j_k}-s_1)\omega_k\} \\ & \times f_0(\omega_1)...f_0(\omega_k)\,d\omega_1...d\omega_k. \end{aligned} \tag{4.26}$$

Let us look at the asymptotic behaviour of the terms in the above sum. Note that all of those $2^{k-1}(k-1)!$ terms behave in the same way asymptotically, hence it is sufficient to consider only one of these terms.

Consider, for example, the following term (which we have chosen only for the sake of simplicity of its treatment to shorten our exposition) :

$$\begin{aligned} & T^{k/2}\frac{1}{(2\pi T)^k}\int_{\mathbb{R}^k}\prod_{i=1}^k\Psi(\lambda_i)\int_{\mathbb{R}^k}\int_{[0,T]^{2k}}e^{-i\Sigma_{j=1}^k(t_j-s_j)\lambda_j}e^{i\Sigma_{j=1}^{k-1}(t_j-s_{j+1})\omega_j+i(t_k-s_1)\omega_k} \\ & \times f_0(\omega_1)...f_0(\omega_k)\,dt_1ds_1...dt_kds_kd\omega_1...d\omega_kd\lambda_1...d\lambda_k \\ = & \frac{T^{k/2}(2\pi)^k}{T^{k-1}}\int_{\mathbb{R}^k}\prod_{i=1}^k\Psi(\lambda_i)\int_{\mathbb{R}^k}\prod_{i=1}^k f_0(\omega_i)\frac{1}{(2\pi)^{2k}T} \\ & \times\int_{[0,T]^{2k}}\exp\{it_1(-\lambda_1+\omega_1)+is_1(\lambda_1-\omega_k)+it_2(-\lambda_2+\omega_2)+is_2(\lambda_2-\omega_1) \\ & +...+it_k(-\lambda_k+\omega_k)+is_k(\lambda_k-\omega_{k-1})\}dt_1ds_1...dt_kds_kd\omega_1...d\omega_kd\lambda_1...d\lambda_k \\ = & \frac{T^{k/2}(2\pi)^k}{T^{k-1}}\int_{\mathbb{R}^k}\prod_{i=1}^k\Psi(\lambda_i) \\ & \times\int_{\mathbb{R}^k}\Phi_T^{2k}(-\lambda_1+\omega_1,\lambda_1-\omega_k,-\lambda_2+\omega_2,\lambda_2-\omega_1,...,-\lambda_k+\omega_k,\lambda_k-\omega_{k-1}) \\ & \times\prod_{i=1}^k f(\omega_i)\,d\omega_1...d\omega_kd\lambda_1...d\lambda_k. \end{aligned} \tag{4.27}$$

Here, for $u_{2k}=-(u_1+...+u_{2k-1})$,

$$\Phi_T^{2k}(u_1,...,u_{2k})=\Phi_T^{2k}(u_1,...,u_{2k-1})=\frac{1}{(2\pi)^kT}\int_{[0,T]^{2k}}\prod_{i=1}^{2k}e^{it_ku_k}dt_1...dt_{2k} \tag{4.28}$$

is the kernel on $\mathbb{R}^{2k-1}$ (see Appendix A). We change the variables in the integral in the last expression in the formula (4.27) by putting

$$
\begin{aligned}
-\lambda_1+\omega_1 &= u_1, -\lambda_1-\omega_k=u_2, -\lambda_2+\omega_2=u_3, \\
\lambda_2-\omega_1 &= u_4, ..., -\lambda_k+\omega_k=u_{2k-1};
\end{aligned}
\tag{4.29}
$$

then we have

$$\lambda_k-\omega_{k-1}=-\left(u_1+...+u_{2k-1}\right).$$

Next, we solve (4.29) to obtain the expressions for the variables λ_i, $i=1,...,k-1$, and ω_i, $i=1,...,k$ in terms of the new variables $u_1,...,u_{2k-1}$ and the variable λ_k. We will not give the exact relationships here, but just notice that these expressions for λ_i, $i=1,...,k-1$, ω_i, $i=1,...,k$ will be of the form $\lambda_i=\lambda_k+l_i\left(u_1,...,u_{2k-1}\right)$, $\omega_i=\lambda_k+m_i\left(u_1,...,u_{2k-1}\right)$, where l_i and m_i are some linear functions of the variables $u_1,...,u_{2k-1}$ (for example, $\lambda_{k-1}=\lambda_k+u_k+u_{k+1}$, $\omega_k=\lambda_k+u_k$, $\omega_{k-1}=\lambda_k+u_k+u_{k+1}+u_1$, and so on). By this change of variables and by using the properties of the kernel $\Phi_T^{2k}\left(u_1,...,u_{2k-1}\right)$, we obtain that (4.27) is equal to the following expression:

$$
\begin{aligned}
&(2\pi)^k T^{-k/2+1}\int_{\mathbb{R}^{2k-1}}\Phi_T^{2k}\left(u_1,...,u_{2k-1}\right) \\
&\times\int_{\mathbb{R}^1}\Psi\left(\lambda_k\right)\Psi\left(\lambda_k+u_k+u_{k+1}\right)...\Psi(\lambda_k+l_k\left(u_1,...,u_{2k-1}\right)) \\
&\times f_0\left(u_k+\lambda_k\right)f_0\left(u_k+\lambda_k+u_{k+1}+u_k\right)... \\
&\times f_0\left(\lambda_k+m_k\left(u_1,...,u_{2k-1}\right)\right)d\lambda_k du_1...du_{2k-1} \\
=\ & O\left(T^{-k/2+1}\right) \text{ as } T\longrightarrow\infty
\end{aligned}
\tag{4.30}
$$

in view of

$$\int_{\mathbb{R}^1}\left\{\Psi\left(\lambda\right)f_0\left(\lambda\right)\right\}^k d\lambda=\int_{\mathbb{R}^1}\left\{\Psi\left(\lambda;\theta_0\right)f\left(\lambda;\theta_0\right)\right\}^k d\lambda<\infty,$$

which can be shown to hold similarly to (4.4) for our choice of the parameters a and b.

The same asymptotics (4.30) is valid for all terms in the sum in the expression for k-th order cumulant (4.26), that is, we have that as $T\longrightarrow\infty$ all cumulants of order $k\geq 3$ of the variable $T^{1/2}Y_T$ tend to zero. Since c_α and c_β are arbitrary, we have shown that the vector $T^{1/2}\nabla_\theta U_T\left(\theta_0\right)$ tends in distribution to a normal random vector with mean 0 and covariance matrix $A\left(\theta_0\right)$ with the elements given by formulae (2.18). This completes the proof of Theorem 2.

Proofs of Theorems 3 and 4.

The proofs are based on the same ideas as those of Theorems 1 and 2 with appropriate modifications due to the multidimensional case. In effect, considering the special case of the Riesz-Bessel motion, we have presented the key ideas of a general theory built on a general set of conditions needed for the convergence in probability of some linear functionals of the periodogram (with specific weight functions) as well as for evaluation of the first and second-order moments and asymptotic distribution of such functionals. These general conditions have been checked for the case of the Riesz-Bessel motion and the weight function $w_{ab}\left(\lambda\right)$, and then have been gathered together as conditions BI - BIX needed to state Theorems 3 and 4, except the condition BVI, which has been introduced to adopt more general arguments for the proof of the uniform (in θ) convergence of our functional $U_T(\theta)$ in the general case.

We only present here some principal ideas. For example, the minimum contrast property for $K(\theta_0;\theta)$, given by (3.7), follows from the relations (3.2) and (3.3) and Jensen's inequality:

$$\begin{aligned}
-K(\theta_0;\theta) &= \int_{\mathbb{R}^n} f(\lambda;\theta_0)\, w(\lambda) \log \frac{\psi(\lambda;\theta)}{\psi(\lambda;\theta_0)} d\lambda \\
&= \sigma^2(\theta_0) \int_{\mathbb{R}^n} \psi(\lambda;\theta_0)\, w(\lambda) \log \frac{\psi(\lambda;\theta)}{\psi(\lambda;\theta_0)} d\lambda \\
&\leq \sigma^2(\theta_0) \log \int_{\mathbb{R}^n} \psi(\lambda;\theta_0)\, w(\lambda)\, d\lambda = 0,
\end{aligned}$$

that is, $K(\theta_0;\theta) \geq 0$; and, moreover, $K(\theta_0;\theta) > 0$ if $\psi(\lambda;\theta_0) \not\equiv \psi(\lambda;\theta)$ for $\theta \neq \theta_0$ almost everywhere with respect to n-dimensional Lebesgue measure. Lemma 1 can be reformulated for the multidimensional case and used to prove that $U_T(\theta) \longrightarrow U(\theta)$ as $T \to \infty$, in probability. To prove, for the general case, that the convergence $U_T(\theta) \longrightarrow U(\theta)$ as $T \to \infty$ holds also uniformly in θ, we apply the following reasoning. By assumption BVI(i), the function $h(\lambda;\theta) = v(\lambda)\log\psi(\lambda;\theta)$ is uniformly continuous in $\mathbb{R}^n \times \Theta$. Denoting by $\eta(\varepsilon)$ its modulus of continuity, we have

$$\sup\{|U_T(\theta_1) - U_T(\theta_2)|, \theta_1, \theta_2 \in \Theta, |\theta_1 - \theta_2| \leq \varepsilon\} \leq \eta(\varepsilon) \int_{\mathbb{R}^n} I_T(\lambda) \frac{w(\lambda)}{v(\lambda)} d\lambda,$$

so the result on consistency will hold if $\int_{\mathbb{R}^n} I_T(\lambda) \frac{w(\lambda)}{v(\lambda)} d\lambda = O_p(1)$. This last relation indeed holds true in view of the multidimensional generalization of Lemma 1, which gives in particular, under the assumption B.VI(ii), the convergence

$$\int_{\mathbb{R}^n} I_T(\lambda) \frac{w(\lambda)}{v(\lambda)} d\lambda \to \int_{\mathbb{R}^n} f_0(\lambda) \frac{w(\lambda)}{v(\lambda)} d\lambda < \infty$$

in probability.

The rest of the proof remains valid; we just need to use some other kernels of Féjer type, of more general form than those used in the proofs for Section 2, namely, the kernels $\widehat{\Phi}_T^{kn}(u_1, ..., u_{k-1})$, $u_i \in \mathbb{R}^n$, $i = 1, ..., k-1$ (see Appendix A).

Appendix A. Multidimensional kernels of the Féjer type

In the proofs, we use the technique based on the results of Bentkus [11], Bentkus [12], Bentkus and Rutkauskas [14] about multidimensional kernels of Féjer type. Consider the following function on $\mathbb{R}^{k-1}$:

$$\begin{aligned}
\Phi_T^k(u_1, ..., u_k) &= \Phi_T^k(u_1, ..., u_{k-1}) = \frac{1}{(2\pi)^{k-1} T} \int_{[0,T]^k} e^{i\Sigma_{j=1}^k t_j u_j} dt_1 ... dt_k \\
&= \frac{1}{(2\pi)^{k-1} T} \prod_{i=1}^k \frac{\sin \frac{T u_j}{2}}{\frac{u_j}{2}},
\end{aligned}$$

where $u_k = -(u_1 + ... + u_{k-1})$, $u_j \in \mathbb{R}^1$, j= $1, ..., k$. The functions $\Phi_T^k(u_1, ..., u_k)$, $k \geq 2$ have the properties which make them similar to the Fejér kernel.

In our exposition we use extensively the following fact.

Proposition 1. *Let the function* $G(u_1, ..., u_k)$, $u_k = -(u_1 + ... + u_{k-1})$ *be bounded and continuous at the point* $(u_1, ..., u_{k-1}) = (0, ..., 0)$. *Then*

$$\lim_{T\longrightarrow\infty}\int_{\mathbb{R}^{k-1}}\Phi_T^k\left(u_1,...,u_{k-1}\right)G\left(u_1,...,u_k\right)du_1...du_{k-1}=G\left(0,...,0\right).$$

We also use the following functions

$$\begin{aligned}\widehat{\Phi}_T^{kn}\left(u_1,...,u_{k-1}\right) &= \frac{1}{\left(2\pi\right)^{n(k-1)}T^n}\int_{[0,T]^{kn}}e^{i\Sigma_{j=1}^k(t_k,u_k)}dt_1...dt_k\\ &= \prod_{i=1}^n\Phi_T^k\left(u_1^{(i)},...,u_{k-1}^{(i)}\right),\end{aligned}$$

where $u_k=-\left(u_1+...+u_{k-1}\right)$, $u_l=\left(u_l^{(1)},...,u_l^{(n)}\right)\in\mathbb{R}^n$, $l=1,...,k$. For the functions $\widehat{\Phi}_T^{kn}\left(u_1,...,u_{k-1}\right)$, $k\geq 2$, the statement analogous to Proposition 1 holds.

Appendix B. Cumulants of Gaussian systems.

This Appendix is based on Hannan ([26]) and Terdik ([40]).

The diagram formula is a basic tool for evaluating the moments of the products of Hermite polynomials of Gaussian random variables. In this paper, we use extensively the formula for the cumulants of products of Gaussian variables. We first introduce some notations and definitions.

Let $m_1,...,m_p$ be given positive integers. An undirected graph Γ with $m_1+...+m_p=M$ vertices is called a diagram of order $(m_1,...,m_p)$ if

a) the set of vertices V of the graph Γ is of the form

$$V=\left\{(1,1),...,(1,m_1),(2,1),...,(2,m_2),...,(p,1),...,(p,m_p)\right\}=\bigcup_{j=1}^p W_j, \tag{B.1}$$

where $W_j=\{(j,l):1\leq l\leq n_j\}$ is the j-th level of the graph $\Gamma, 1\leq j\leq p$;

b) each vertex is at most of degree 1, that is, met by at most one edge;

c) if vertices (j_1,i_1) and (j_2,i_2) are joined by an edge $\omega=((j_1,i_1),(j_2,i_2))$, then $j_1\neq j_2$, that is, the edges of the graph Γ can connect only different levels.

Let $\Gamma\left(m_1,...,m_p\right)$ denote the set of diagrams of order $\left(m_1,...,m_p\right)$. Denote by $\mathcal{K}\left(\gamma\right)$ the set of edges of a diagram $\gamma\in\Gamma\left(m_1,...,m_p\right)$. With each element $v\in V$, we can associate the integer denoting the position at which v appears at the list (B.1). Thus the position of $(1,1)$ is 1, the position of $(1,2)$ is 2 and so on. The position of the last vertex (p,m_p) is M. Each edge $\omega=\left(\left(j_1,i_1\right),\left(j_2,i_2\right)\right)\in\mathcal{K}\left(\gamma\right)$ can also be thought of as $\omega=\left(k_1,k_2,\right)$ where k_1 is the position of the vertex (j_1,i_1) and k_2 is the position of the vertex (j_2,i_2) in the list (B.1) A diagram γ is called *complete* if each of its vertices is met by an edge, that is, there exists no isolated vertices. In such a case, the number of edges in γ is $|\mathcal{K}\left(\gamma\right)|=M/2$. A diagram is called *closed* if the set of its levels $\{W_j,j=1,...,p\}$ cannot be split into two subsets connected by no edge.

Let X be M-dimensional Gaussian vector with zero mean and covariance $Cov\left(X_s,X_t\right)=\sigma\left(s,t\right)$, $s,t=1,...,M$, and let positive numbers $m_1,...,m_p$ be given such that $m_1+...+m_p=M$; denote $M_i=m_1+...+m_i$, $i=1,...,p$, $M_p=M$. We have the formula

$$Cum\left(\prod_{i=1}^{M_1}X_i,\prod_{i=M_1+1}^{M_2}X_i,...,\prod_{i=M_{p-1}+1}^{M}X_i\right)=\sum_{\gamma\in\Gamma(m_1,...,m_p)}\prod_{(t_i,t_j)\in\mathcal{K}(\gamma)}\sigma\left(t_i,t_j\right),$$

where summation is taken over all complete closed diagrams of order $(m_1, ..., m_p)$, $\mathcal{K}(\gamma)$ is the set of edges of the diagrams γ.

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